Skew bracoids

Isabel Martin-Lyons

joint work with Paul Truman

Keele University, UK

International Workshop on Non-Associative Algebras, Lecce

1st of March 2024

Outline

Outline

The skew bracoid

Definition (M-L and Truman, 2024)

A *skew bracoid* is a quintuple $(G, \circ, N, +, \odot)$ with $(G, \circ)$ and $(N, +)$ groups and $\odot$ a transitive action of G on N satisfying

$$g \odot (\eta + \mu) = (g \odot \eta) - (g \odot e_N) + (g \odot \mu) \tag{1}$$

for all $g \in G$ and all $\eta, \mu \in N$.

- We refer to
  - (1) as the *skew bracoid relation*;
  - $(G, \circ)$ as the acting or multiplicative group;
  - and $(N, +)$ as the additive group, though we do not assume $+$ is abelian (and do not always write it additively).
- We will frequently write (G, N) for $(G, \circ, N, +, \odot)$.

Examples

Examples

- Recall that a *skew brace* is a triple $(G, +, \circ)$ with $(G, +)$ and $(G, \circ)$ groups and
$$g \circ (h + h') = g \circ h - g + g \circ h',$$
for all $g, h, h' \in G$. Any skew brace $(G, +, \circ)$ can be thought of as a skew bracoid $(G, \circ, G, +, \odot)$ with $g \odot h := g \circ h$.
- Let $d, n \in \mathbb{N}$ such that $d|n$. Take $N = \langle \eta \rangle \cong C_d$ and $G = \langle r, s \mid r^n = s^2 = e, srs^{-1} = r^{-1} \rangle \cong D_n$. Then we get a skew bracoid (G, N) using the action $\odot$ given by
$$r^i s^j \odot \eta^k = \eta^{i+(-1)^j k}.$$

The λ-function

Definition/Proposition

Given a skew bracoid $(G, \circ, N, +, \odot)$, we define the map $\lambda : (G, \circ) \to \text{Perm}(N, +)$, sending g to λ_g, by

$$\lambda_g(\eta) = -(g \odot e_N) + (g \odot \eta),$$

for $g \in G$ and $\eta \in N$.

Then λ is in fact a homomorphism, with image in $\text{Aut}(N, +)$. We call this map the *λ-function* of the skew bracoid.

Examples

- If we are actually in a skew brace $(G, \circ, G, +, \odot)$ then

$$\begin{aligned}\lambda_g(h) &= -(g \odot e) + (g \odot h) \\ &= -(g \circ e) + (g \circ h) \\ &= -g + g \circ h,\end{aligned}$$

  which we recall agrees with the typical λ-function of the skew brace.

- In (D_n, C_d) we have

$$\begin{aligned}\lambda_{r^i s^j}(\eta^k) &= (r^i s^j \odot e_N)^{-1}(r^i s^j \odot \eta^k) \\ &= \eta^{-i}\eta^{i+(-1)^j k} \\ &= \eta^{(-1)^j k}.\end{aligned}$$

As a surjective 1-cocycle

Let $\pi : G \to N$ be the map taking g to $g \odot e_N$, then as $\odot$ is a transitive action π is a surjective. We have

$$\begin{aligned}\pi(g) + \lambda_g(\pi(h)) &= (g \odot e) - (g \odot e) + (g \odot (h \odot e)) \\ &= gh \odot e \\ &= \pi(gh)\end{aligned}$$

for all $g, h \in G$, i.e. π is then a surjective 1-cocycle for the action of G on N by automorphisms via λ.

Conversely, given G and N groups, a homomorphism $\lambda : G \to \mathrm{Aut}(N)$ and a surjective 1-cocycle $\pi : G \to N$, we can define an action of G on N via $g \odot \eta = \pi(g) + \lambda_g(\eta)$. With this action (G, N) is a skew bracoid.

In the Holomorph

Let $(N, +)$ be a group and A be a transitive subgroup of $\mathrm{Hol}(N, +) = N \rtimes \mathrm{Aut}(N)$. Then (A, N) is a skew bracoid with the obvious action of A on N.

Conversely, given a skew bracoid (G, N) we can map G into $\mathrm{Hol}(N)$ via $g \mapsto (g \odot e_N)\lambda_g$. The image of this map is then isomorphic to the quotient $G/\ker(\odot)$, where $\ker(\odot) = \{g \in G \mid g \odot \eta = \eta \text{ for all } \eta \in N\}$.

Example

Consider $(G, N) \cong (D_n, C_d)$. Writing ι for inversion in N, we have $r^i s^j \mapsto \eta^i \iota^j$. Hence the image of G in $\mathrm{Hol}(N)$ is $N \rtimes \langle \iota \rangle$. This is isomorphic to G itself precisely when $d = n$.

Outline

Ideals of a Skew Brace

Given a skew brace $(G, +, \circ)$, recall $\lambda_g(h) = -g + g \circ h$.

Definition

Let $(G, +, \circ)$ be a skew brace. A subgroup I of $(G, +)$ is

- a *left ideal* if it is closed under λ_G, this gives for free that I is a subgroup of $(G, \circ)$ and that $g + I = g \circ I$ for all $g \in G$;
- a *strong left ideal* if I is additionally normal in $(G, +)$;
- an *ideal* if I is normal in both $(G, +)$ and $(G, \circ)$.

As one would expect, to take a quotient we want an ideal of our skew brace. Since cosets agree, we may simply take the quotient in the group sense and recover a skew brace.

A Partial Quotient

Instead let us attempt to quotient by merely a strong left ideal.

- Take a skew brace $(G, +, \circ)$ and a strong left ideal I.
- Recall I is normal in G with respect to $+$, so G/I makes sense as a group with $+$.
- We can also think of G/I as a $\circ$-coset space and let G act (transitively) on G/I by left translation of cosets.
- Writing $\odot$ for this action, for all $g, h, h' \in G$ we have

$$\begin{aligned} g \odot (hI + h'I) &= g \odot (h + h')I \\ &= (g \circ (h + h'))I \\ &= (g \circ h - g + g \circ h')I \\ &= (g \circ h)I - gI + (g \circ h')I \\ &= (g \odot hI) - (g \odot e_G I) + (g \odot h'I), \end{aligned}$$

so that $(G, \circ, G/I, +, \odot)$ is a skew bracoid.

Our example

Example

Take $G = \langle r, s \rangle \cong D_n$ as before. We can define a second binary operation on G by $r^i s^j \cdot r^k s^\ell := r^{i+k} s^{j+\ell}$. Then $(G, \cdot, \circ)$ is a skew brace with $(G, \circ) \cong D_n$, $(G, \cdot) \cong C_n \times C_2$ and λ-function given by

$$\lambda_{r^i s^j}(r^k s^\ell) = r^{-i} s^{-j} \cdot (r^i s^j \circ r^k s^\ell) = r^{(-1)^j k} s^\ell.$$

Let d be some divisor of n and $I = \langle r^d, s \rangle$. It is straightforward to check that I is strong left ideal of $(G, \cdot, \circ)$ and $G/I = \langle rI \rangle \cong C_d$, so that $(G, G/I)$ is our familiar skew bracoid.

A Natural Question

Question

Do **all** skew bracoids arise as a quotient of a skew brace by a strong left ideal?

Thanks to Byott we have an example of a skew bracoid $(G, N) \cong (GL_3(\mathbb{F}_2), C_2 \times C_2 \times C_2)$ for which the only additive operation on G giving $(G, +, \circ)$ a skew brace has $(G, +)$ simple. This means there are no strong left ideals to quotient by, so this skew brace cannot directly arise as a quotient as outlined.

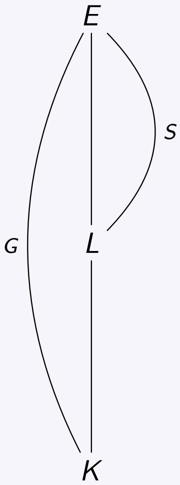
A slightly less natural question

However, is there some larger $(H, +, \circ)$ with strong left ideal I, such that $(H/\ker(\odot), \circ, H/I, +, \odot)$ is isomorphic to (G, N)?

Outline

Setting

Let E/K be a finite Galois extension of fields with L some intermediate field, so that L/K is separable but not necessarily Galois. Write $(G, \circ) = \mathrm{Gal}(E/K)$ and $S = \mathrm{Gal}(E/L)$.



Definition

A *Hopf-Galois structure* on L/K is a K-Hopf algebra H together with an action $*$ of H on L such that

- L is a H-module algebra;
- the map $j : L \otimes_K H \to \mathrm{End}_K(L)$ given by $j(x \otimes h) : y \mapsto x(h * y)$ is an isomorphism.

A (Very) Brief History of Hopf-Galois Theory

- Chase and Sweedler [1969] introduce the study of Hopf-Galois theory with a view to inseparable extensions of fields and ramified extensions of rings.
- Greither and Pareigis [1987] characterise Hopf-Galois structures on separable extensions using certain subgroups of $\mathrm{Perm}(G/S)$.
- Byott [1996], following an observation by Childs [1989], gave a further correspondence between this permutation setting and the holomorph.
- Bachiller [2016] sets out a correspondence between braces and subgroups of the holomorph and notes its relevance to Hopf-Galois theory.
- Byott and Vendramin [2018] make the connection between Hopf-Galois structures on Galois extensions and skew braces via their mutual connection to regular subgroups of the holomorph.
- Stefanello and Trappeniers [2023] realign the correspondence, making it a bijection and giving more qualitative results.

A Quick Note

Fact

Any skew bracoid (G, N) can be written in the form $(G, G/S)$ where $S = \mathrm{Stab}_G(e_N)$. We have a bijection

$$gS \longleftrightarrow g \odot e_N$$

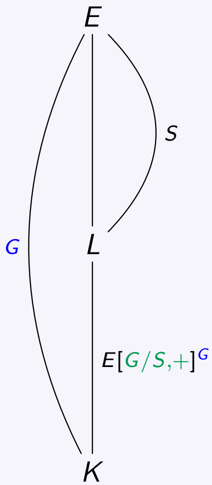
which we can use to transport the operation in N to the coset space G/S; notice that the identity coset, $e_G S$, is then the identity in this group. Under this bijection the action of G on G/S becomes left translation of cosets via the operation in G.

The correspondence

Theorem (M-L and Truman, 2024)

There is a bijective correspondence between

- Hopf-Galois structures on L/K and
- operations $+$ such that $(G, \circ, G/S, +, \odot)$ forms a skew bracoid, with $e_G S = e_{G/S}$, and where $\odot$ is left translation of cosets via $\circ$.



Explicitly, the Hopf-Galois structure coming from $(G, G/S)$ is $E[G/S, +]^G$ with action on L given by

$$\left(\sum_{gS\in G/S} c_{gS}\, gS\right)[t] = \sum_{gS\in G/S} c_{gS}\, gS[t].$$

The Hopf-Galois Correspondence

Theorem (Greither and Pareigis, 1987)

Suppose H is a Hopf-Galois stucture on L/K. For a K-Hopf subalgebra H' of H define

$$\mathrm{Fix}(H') = L^{H'} = \{x \in L \mid h * x = \epsilon(h)x \text{ for all } h \in H'\}.$$

Then the map

$$\mathrm{Fix} : \{K\text{-Hopf subalgebras of } H\} \to \{\text{intermediate fields of } L/K\}$$

is injective and inclusion reversing.

But when is it also surjective?

Ideals of a Skew Brace

Given a skew brace $(G, +, \circ)$, recall $\lambda_g(h) = -g + g \circ h$.

Definition

Let $(G, +, \circ)$ be a skew brace. A subgroup I of $(G, +)$ is

- a *left ideal* if it is closed under λ_G, this gives for free that I is a subgroup of $(G, \circ)$ and that $g + I = g \circ I$ for all $g \in G$;
- a *strong left ideal* if I is additionally normal in $(G, +)$;
- an *ideal* if I is normal in both $(G, +)$ and $(G, \circ)$.

As one would expect, to take a quotient we want an ideal of our skew brace. Since cosets agree, we may simply take the quotient in the group sense and recover a skew brace.

Ideals of a Skew Bracoid and the HGC

Let $(G, G/S)$ be a skew bracoid and G' be a subgroup of G containing S.

Definition (For our purposes)

We say $G \odot e_{G/S}$ is

- a *left ideal* of $(G, G/S)$ if it is closed under λ_G;
- an *ideal* of $(G, G/S)$ if $G' \odot e_{G/S}$ is additionally normal in G/S.

Theorem (M-L and Truman, 2024)

Let $E[G/S]^G$ be a HGS on L/K, corresponding to $(G, G/S)$. Take G' as above so that $L^{G'}$ is an intermediate field of L/K. Then

- $L^{G'}$ occurs in the image of the HGC for $E[G/S]^G \iff G' \odot e_{G/S}$ is a left ideal of $(G, G/S)$.
- we can form a quotient structure on $L^{G'}/K \iff G' \odot e_{G/S}$ is an ideal.

The Hopf-Galois Correspondence for Our Example

Example

Take $(G, G/S) \cong (D_n, C_d)$.

- Any subgroup G' of G containing $S = \langle r^d, s\rangle$ will be of the form $\langle r^f, s\rangle$ for some $f|d$.
- Then $G' \odot e_G S = \{r^{if} S \mid 0 \leq i < f\}$.
- Further $\lambda_{r^i s^j}(r^{kf} S) = r^{(-1)^j kf} S \in G' \odot e_G S$ so $G' \odot e_G S$ is closed under λ_G.
- From this or by inspection we see $G' \odot e_G S$ is a subgroup of G/S.
- Hence $L^{G'}$ occurs in the image of the HGC for the Hopf-Galois structure $E[G/S]^G$ for all G' so the HGC is surjective in this case.
- Since G/S is abelian each $G' \odot e_G S$ is additionally normal so we can form a quotient Hopf-Galois structure on $L^{G'}/K$ again for every G'.

Outline

Open Questions

- Stefanello and Trappeniers give various families of Hopf-Galois structures for which the Hopf-Galois correspondence is surjective, can we find such families in the separable case? We can show that so called almost classical structures have this property from the skew bracoid perspective but this was well known from Hopf-Galois theory alone.
- We would like some notions of products of skew bracoids. So far we have an idea of an induced skew bracoid, coming from Hopf-Galois theory, but only vague ideas of what a semi-direct or matched product should entail.
- Do skew bracoids have anything to do with solutions to the Yang-Baxter equation? Yes! See [Colazzo, Koch, M-L, and Truman, hopefully 2024?].

Thank you for your attention!

References I

I M-L and Paul J. Truman. Skew bracoids. *Journal of Algebra*, 638:751–787, 2024. ISSN 0021-8693. doi: https://doi.org/10.1016/j.jalgebra.2023.10.005. URL https://www.sciencedirect.com/science/article/pii/S0021869323005136.

Stephen U. Chase and Moss E. Sweedler. *Hopf algebras and Galois theory*, pages 52–83. Springer Berlin Heidelberg, Berlin, Heidelberg, 1969. ISBN 978-3-540-36134-3. doi: 10.1007/BFb0101435. URL https://doi.org/10.1007/BFb0101435.

Cornelius Greither and Bodo Pareigis. Hopf Galois theory for separable field extensions. *Journal of Algebra*, 106(1):239–258, 1987. ISSN 0021-8693. doi: https://doi.org/10.1016/0021-8693(87)90029-9. URL https://www.sciencedirect.com/science/article/pii/0021869387900299.

References II

Nigel P. Byott. Uniqueness of Hopf Galois structure for separable field extensions. *Communications in Algebra*, 24(10):3217–3228, 1996. doi: 10.1080/00927879608825743. URL https://doi.org/10.1080/00927879608825743.

Lindsay N. Childs. On the Hopf Galois theory for separable field extensions. *Communications in Algebra*, 17(4):809–825, 1989. doi: 10.1080/00927878908823760. URL https://doi.org/10.1080/00927878908823760.

David Bachiller. Counterexample to a conjecture about braces. *Journal of Algebra*, 453:160–176, 2016. ISSN 0021-8693. doi: https://doi.org/10.1016/j.jalgebra.2016.01.011. URL https://www.sciencedirect.com/science/article/pii/S0021869316000351.

References III

L. Guarnieri and L. Vendramin. Skew braces and the Yang–Baxter equation. *Mathematics of Computation*, 86(307):2519–2534, 2017. ISSN 1088-6842. doi: 10.1090/mcom/3161. URL http://dx.doi.org/10.1090/mcom/3161.

Agata Smoktunowicz and Leandro Vendramin. On skew braces (with an appendix by N. Byott and L. Vendramin). *Journal of Combinatorial Algebra*, 2(1):47–86, February 2018. ISSN 2415-6302. doi: 10.4171/JCA/2-1-3.

Lorenzo Stefanello and Senne Trappeniers. On the connection between Hopf–Galois structures and skew braces. *Bulletin of the London Mathematical Society*, 2023. doi: https://doi.org/10.1112/blms.12815. URL https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/blms.12815.